CEEPR Center for Energy and Environmental Policy Research

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by
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09-018 September 2009

A Joint Center of the Department of Economics,
MIT Energy Initiative, and Sloan School of Management

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Abstract

Equilibrium models have been proposed in literature with the aim of describing the evolution of the price of emission permits. This paper derives first estimation methods for the calibration of three competing equilibrium models. Second, it demonstrates how their reduced-form versions are inter-related. Third, by means of calibration to historical data, it is shown how these reduced-form models perform in the current price-evolution framework also with respect to standard continuous time stochastic models.

Key words: CO_2 emission allowances, Equilibrium model, Model calibration.

JEL Classifications: C02, C61, C63, C65, G13.

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Acknowledgements: We would like to thank Rüdiger Kiesel, Marc Chesney, Matthew Nicholson for their valuable comments and discussions. This paper was partly written while Taschini was Research Visiting Fellow at the MIT Center for Energy and Environmental Policy Research, Cambridge, MA - USA. Taschini gratefully acknowledges financial support from LSE. The usual disclaimers apply.

1 Introduction

For more than two decades, environmental law and regulation was dominated by command-and-control approaches typically either mandated pollution control technologies or inflexible discharge standards. In the 1980s policy makers increasingly explored market-based environmental policy instruments. Such mechanisms should provide economic incentives for firms and individuals to carry out cost-effective pollution control. In particular, in a market-based system the theory is that participants trade permits thereby minimizing the cost of pollution control to society. The source of these cost savings is the capacity of economic instruments to take advantage of the large differentials abatement costs across polluters. [1] In a cap-and-trade system, regulators set a target level for emissions (i.e. the cap) and issue permits which are allocated according to different criteria (auctioning, grandfathering, etc.) to the installations participating in the program. To enforce the cap, a penalty is levied for each unit of pollutant emitted outside the limits of a given compliance period. Firms may either reduce their own pollution or purchase emission permits in order to ensure compliance. This transfer of permits by trading is the core principle leading to the minimization of the costs caused by regulation; firms that can easily reduce emissions will do so, while those that cannot buy permits. Cap-and-trade systems continue today to be at the center of actions linked with global climate change. In 2005, in an effort to meet targets under the Kyoto Protocol, European policy makers launched the so-called European Emission Trading Scheme (EU ETS). Most recently, cap-and-trade systems have been discussed as a possible means to reduce carbon dioxide and other greenhouse gas emissions in the U.S.

Given the prevalence of cap-and-trade schemes, a clear understanding of the carbon pricing mechanism is obvious. Only a handful of papers in the literature are devoted to permit pricing and we briefly review those related to our paper. One of the first references to market-based techniques for dealing with pollution problems can be found in the seminal works of Coase [8] and Dales [9]. Based on such an idea, Montgomery [14] provides, in a deterministic setting, a rigorous theoretical justification of how a market-based approach leads to the efficient allocation of abatement costs across various pollution sources. Recently, in an effort to bridge the gap between theory and observed market-price behavior, an increasing number of empirical studies have investigated the historical time series of the price of emission permits. In the context of the first EU ETS phase (2005-2007) the following classes of processes have been applied to the permit price series: jump-diffusion mod-

[1] We refer to Baumol and Oates [2] for a complete discussion on market-based policy measures, and to Taschini [17] for an introductory review on fundamental concepts in environmental economics.

els (Wagner [19], Daskalakis et al. [10]), GARCH-models (Benz and Trück [3] and Wagner [19]), regime-switching models (Wagner [19], Benz and Trück [3]), Mix-Normal GARCH-models (Paolella and Taschini [15]) and two-factor models (Cetin and Verschuere [6]). Other authors support the argument that the permit price responds to macroeconomic fundamentals and try to explain the price evolution of emission permits in terms of electricity, gas, oil and coal prices and weather effects (cf. Hintermann [12] and Mansanet-Bataller et al. [13]).

A theoretical strand of literature evolved recently describing the price dynamics of emission permits by tailor-made stochastic equilibrium models. Allowing for stochastic production costs, revenues from selling produced goods and emission quantities, Carmona et al. [4] showed in a general setting that the price of emission permits equals the discounted penalty multiplied by the probability of the event of shortage (i.e. the aggregated cumulative emissions exceed total number of permits). The models of Chesney and Taschini [7] and Grüll and Kiesel [11] specify the process for the cumulative emissions in the framework of Carmona et al. [4] by assuming that the firms' emission rate follows a geometric Brownian motion. This means that the cumulative emissions are described by the integral over a geometric Brownian motion for which no closed-form density is available. The models of Chesney and Taschini [7] and Grüll and Kiesel [11] differ in the way the cumulative emissions are approximated. The linear approximation approach of Chesney and Taschini [7] has the shortcoming that the moments of the approximated cumulative emissions do not match the true ones. Grüll and Kiesel [11] overcome this problem by applying a moment matching approach.
However, so far this type of literature including the above papers focused on showing theoretical properties of emission trading systems rather than calibrate the model parameters to historical time series. Carmona et al. [4] analyze the effect of windfall profits, Chesney and Taschini [7] investigate the effect of asymmetric information on the permit price and Grüll and Kiesel [11] provide a theoretical sound discussion about the permit price slump in 2006 in the EU ETS.

With the objective to provide tractable pricing models for options on emission permits, Carmona and Hinz [5] were the first to address the complexity of the calibration of the equilibrium model of Carmona et al. [4]. The authors introduce a simple risk-neutral reduced-form model for the price of emission permits and calibrate it to historical data. Our contribution extends Carmona and Hinz [5] efforts by deriving estimation methods for the calibration to real data of those competing equilibrium models introduced in this paper. Furthermore, for the first time in the literature, we compare in-sample performances of reduced-form models including into the analysis standard continuous-time stochastic processes (i.e. geometric Brownian motion and normal Inverse Gaussian). Finally, we prove the existing relationship between the reduced-form model of Carmona and Hinz and the full-model of Chesney and Taschini.

The present paper is organized as follows. Section 2 introduces the equilibrium model of Chesney and Taschini [7], its modification proposed by Grüll and Kiesel [11], and the model of Carmona and Hinz [5]. In section 3 we propose different approximation approaches with the aim of obtaining a handy stochastic differential equation which is flexible enough to describe the historical price evolution of emission permits. We also show the analytical relationship between the models of Chesney and Taschini [7] and Carmona and Hinz [5]. Section 4 investigates historical model calibrations and compares reduced-form models and standard continuous-time stochastic models performances. Section 5 concludes.

2 Equilibrium Models

In this section we introduce the full equilibrium model of Chesney and Taschini [7] (hereafter CT), its modification proposed by Grüll and Kiesel [11] (hereafter GK), and the full equilibrium model of Carmona et al. [4] (hereafter CFHP). Also, we present the reduced-form model of Carmona and Hinz [5] (hereafter CH). For a comprehensive overview of other recent attempts at developing valid price models for emission permits we refer to Taschini [17].

Both the models of CT and GK assume that the firms' pollution emission rate Q_t follows a geometric Brownian motion

$$dQ_t = Q_t[\mu dt + \sigma dW_t].$$

Therefore, the cumulative emissions in $[0, t]$ are given by

$$q_{[0,t]} = \int_0^t Q_s ds.$$

Let us also introduce P as the penalty that has to be paid for each emission unit that is not covered by a permit at the compliance date T. Also, N is the total amount of permits allocated by the policy regulator to relevant companies, i.e. the cap.

The models of CT and GK differ in the way the cumulative emissions are approximated. CT approximate the cumulative emissions linearly

$$\begin{aligned} q_{[t_1,t_2]} \approx \tilde{q}^{CT}_{[t_1,t_2]} &= Q_{t_2}(t_2 - t_1) \\ &= Q_{t_1} \exp\left\{ \ln(t_2 - t_1) + \left(\mu - \frac{\sigma^2}{2}\right)(t_2 - t_1) + \sigma\sqrt{t_2 - t_1} Z \right\}, \end{aligned}$$

where Z is a standard normally distributed random variable. GK uses a moment matching approach for approximation that yields

$$q_{[t_1,t_2]} \approx \tilde{q}^{GK}_{[t_1,t_2]} = Q_{t_1} \exp\left\{ \ln\left(\frac{\alpha^2_{t_2-t_1}}{\sqrt{2\beta_{t_2-t_1}}}\right) + \sqrt{\ln\left(\frac{2\beta_{t_2-t_1}}{\alpha^2_{t_2-t_1}}\right)} Z \right\},$$

where

$$\alpha_{t_2-t_1} = \begin{cases} \frac{1}{\mu}\left(e^{\mu(t_2-t_1)} - 1\right) & \text{if } \mu \neq 0 \\ t_2 - t_1 & \text{if } \mu = 0 \end{cases} \tag{1}$$

$$\beta_{t_2-t_1} = \begin{cases} \frac{\mu e^{(2\mu+\sigma^2)(t_2-t_1)} + \mu + \sigma^2 - \left(2\mu+\sigma^2\right) e^{\mu(t_2-t_1)}}{\mu(\mu+\sigma^2)(2\mu+\sigma^2)} & \text{if } \mu \neq 0 \\ \frac{1}{\sigma^4}\left(e^{\sigma^2(t_2-t_1)} - 1\right) & \text{if } \mu = 0 \end{cases} \tag{2}$$

CFHP prove in a general setting that the futures price of emission permits at time t is given by

$$F(t,T) = P \cdot \mathbb{P}\left(q_{[0,T]} > N | \mathcal{F}_t\right), \tag{3}$$

where $q_{[0,T]}$ is the random variable that denotes the aggregated cumulative emissions of all relevant companies at time T.

Addressing the problem of pricing options contracts on emission permits, in a recent paper CH develop a reduced-form model and propose a risk-neutral price dynamics of emission permits. Under the risk-neutral measure $\mathbb{Q}$, the futures permit price $F(t,T)$ is modeled as

$$F(t,T) = P \cdot \mathbb{Q}\left(\Gamma_0 \exp\left\{\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T \sigma_s^2 ds\right\} > 1 \mid \mathcal{F}_t\right),$$

where $\Gamma_0 \in (0,\infty)$, and $\sigma(\cdot)$ is a continuous square integrable deterministic function. CH prove that the futures permit price under the historical measure $\mathbb{P}$ for some fixed $h \in \mathbb{R}$, is given by

$$\begin{aligned} F(t,T) &= P \cdot \mathbb{P}\left(\Gamma_0 \exp\left\{\int_0^T \sigma_s(dW_s + h ds) - \frac{1}{2}\int_0^T \sigma_s^2 ds\right\} > 1 \mid \mathcal{F}_t\right) \\ &= P \cdot \mathbb{P}\left(\Gamma_0 \exp\left\{\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T (\sigma_s^2 - 2h\sigma_s) ds\right\} > 1 \mid \mathcal{F}_t\right). \end{aligned} \tag{4}$$

Comparing Equations (3) and (4), the aggregated cumulative emissions normalized with respect to the cap are described by the following process

$$\frac{q_{[0,T]}}{N} = \Gamma_0 \exp\left\{\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T (\sigma_s^2 - 2h\sigma_s) ds\right\}. \tag{5}$$

It is important to notice that the cumulative emissions under such a specification do not satisfy two important (and quite natural) properties of fund pollutant.[2] They are neither additive in time (i.e. $q_{[0,T]} \neq q_{[0,t]} + q_{[t,T]}$ for $t < T$) nor do they strictly increase over time. However, this assumption makes computations in CH much easier and yields a tractable option pricing model. Following the definition of CH, we call this type of simplified equilibrium models reduced-form models. Based on Equation (5), under the risk-neutral measure the futures price of emission permits $F(t,T)$ divided by the penalty P satisfies the following stochastic differential equation (SDE)

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{\frac{\beta}{T-t}}dW_t,$$

where $a_t = \frac{F(t,T)}{P}$. Under the historical measure, a_t satisfies the following SDE

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{\frac{\beta}{T-t}}(dW_t + hdt) \tag{6}$$

$$= \Phi'(\Phi^{-1}(a_t))\left[\sqrt{\frac{\beta}{T-t}}dW_t + h\sqrt{\frac{\beta}{T-t}}dt\right]. \tag{7}$$

By means of discretization of this SDE and using the closed-form maximum-likelihood estimation reported by CH, the parameters of this reduced-form model can be estimated.

3 Estimation methods for full models

The aim of this section is to determine sufficiently flexible dynamics of the price of emission permits to be then calibrated using market data. We propose and discuss three different approaches for approximating the model of CT and one for approximating the model of GK, respectively. Each approximation allows us to derive possible estimation methods. Furthermore, we employ them to investigate how the models introduced in Section 2 are related to each other.

The derivation of the price dynamics is done in two steps. First, we derive the theoretical price of emission permits at time t in the framework of CT, assuming that we know the emission rate at time t and the aggregated cumulative emissions until t (cf. Lemma 1). The SDE for the price dynamics is obtained in a second step by treating the emission rate and cumulative emissions in

[2] Cap-and-trade schemes are typically implemented to curb pollutants that need total volume control because of the existence of a threshold in the flow or stock of them - see Tietenberg [18].

Lemma 1 as random variables.

Lemma 1 (Permit price in the model of Chesney and Taschini)**.**
The time-t permit price divided by the penalty is given by

$$a_t = \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

In particular, we have

$$a_0 = \Phi\left(\frac{-\ln\left(\frac{N}{T\cdot Q_0}\right) + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

Proof :
Approximating the cumulative emissions linearly, i.e. $q_{[t,T]} \approx (T-t)\cdot Q_T$, and letting $Z \sim N(0,1)$ we get

$$\begin{aligned}
a_t =\ & \mathbb{P}\left((T-t)\cdot Q_T > N - q_{[0,t]}|\mathcal{F}_t\right) \\
=\ & \mathbb{P}\left((T-t)\cdot Q_t \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}Z\right\} > N - q_{[0,t]}|\mathcal{F}_t\right) \\
=\ & \mathbb{P}\left(\exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}Z\right\} > \frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}|\mathcal{F}_t\right) \\
=\ & \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

$\diamondsuit$

Deriving an SDE for the permit price in Lemma 1 yields a very complicated expression that cannot be used for model calibration in practice. Therefore, we propose three different approaches for the approximation of the random variable $\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}$ in Lemma 1. By means of the first approximation approach, we show the relationship between the models of CT and CH. The second and third approximation approaches are proposed for the purpose of getting a more tractable model calibration.

Approximation 1: Assume that the emission rate follows a geometric Brownian motion with a deterministic time-dependent drift μ_s and a diffusion coefficient σ_s. Let us define the "longness" of the market as the number of remaining permits divided by the emissions in the remaining time period given the current emission rate, i.e. $\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}$. Values greater (less) than 1 correspond to a situation where the emission market is long (short) in permits. Assume that this "longness" follows a geometric Brownian motion

$$\frac{N - q_{[0,t]}}{(T-t) \cdot Q_t} \approx \frac{N}{T \cdot Q_0} \exp\left\{ \int_0^t \left(\tilde{\mu}_s - \frac{\tilde{\sigma}_s^2}{2} \right) ds + \int_0^t \tilde{\sigma}_s dW_s \right\}$$

where $\tilde{\mu}_s$ and $\tilde{\sigma}_s$ are deterministic functions.

Theorem 2 (SDE for approximation approach 1)**.**

(a) Let Q_t be an emission rate with time-dependent drift and volatility, i.e.

$$dQ_t = Q_t[\mu_t dt + \sigma_t dW_t]$$

for deterministic functions μ_t and σ_t.
Then in the model of CT the permit price divided by the penalty is given by

$$a_t = \Phi \left(\frac{-\ln\left(\frac{N - q_{[0,t]}}{(T-t) \cdot Q_t}\right) + \int_t^T \left(\mu_s - \frac{\sigma_s^2}{2} \right) ds}{\sqrt{\int_t^T \sigma_s^2 ds}} \right).$$

(b) Approximate the "longness" by

$$\frac{N - q_{[0,t]}}{(T-t) \cdot Q_t} \approx \frac{N}{T \cdot Q_0} \exp\left\{ \int_0^t \left(\tilde{\mu}_s - \frac{\tilde{\sigma}_s^2}{2} \right) ds + \int_0^t \tilde{\sigma}_s dW_s \right\},$$

where $\tilde{\mu}_s$ and $\tilde{\sigma}_s$ are deterministic functions. Then the dynamics of the permit price in the model of CT are given by

$$da_t = -\frac{\Phi'\left(\Phi^{-1}(a_t)\right)}{\sqrt{\int_t^T \sigma_s^2 ds}} \left[\left(\tilde{\mu}_t + \mu_t - \frac{\tilde{\sigma}_t^2}{2} - \frac{\sigma_t^2}{2} + \frac{1}{2} \frac{\tilde{\sigma}_t^2 - \sigma_t^2}{\sqrt{\int_t^T \sigma_s^2 ds}} \Phi^{-1}(a_t) \right) dt + \tilde{\sigma}_t dW_t \right].$$

(c) The model of CT with time-dependent emission rate can be transformed into the model of CH by setting

$$\begin{aligned} \tilde{\sigma}_t &= -\sigma_t = -\sqrt{\beta(T-t)^{\beta-1}}, \\ \tilde{\mu}_t &= -\mu_t + \sigma_t(\sigma_t - h). \end{aligned}$$

Proof :
(a) Follows directly from Lemma 1 and $Q_t = Q_0 \exp\left\{ \int_0^t \left(\mu_s - \frac{\sigma_s^2}{2} \right) ds + \int_0^t \sigma_s dW_s \right\}$.

(b) Approximation 2 and part (a) yield

$$
\begin{aligned}
a_t = &\ \Phi\left(\frac{-\ln\left(\frac{N}{T\cdot Q_0}\exp\left\{\int_0^t\left(\tilde{\mu}_s-\frac{\tilde{\sigma}_s^2}{2}\right)ds+\int_0^t\tilde{\sigma}_s dW_s\right\}\right)+\int_t^T\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds}{\sqrt{\int_t^T\sigma_s^2 ds}}\right)\\
= &\ \Phi\left(\frac{-\ln\left(\frac{N}{T\cdot Q_0}\right)-\int_0^t\left(\tilde{\mu}_s-\frac{\tilde{\sigma}_s^2}{2}\right)ds-\int_0^t\tilde{\sigma}_s dW_s+\int_t^T\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds}{\sqrt{\int_t^T\sigma_s^2 ds}}\right)\\
= &\ \Phi\left(\frac{-\ln\left(\frac{N}{T\cdot Q_0}\right)-\int_0^t\left(\tilde{\mu}_s-\frac{\tilde{\sigma}_s^2}{2}\right)ds-\int_0^t\tilde{\sigma}_s dW_s+\int_0^T\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds-\int_0^t\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds}{\sqrt{\int_t^T\sigma_s^2 ds}}\right)\\
= &\ \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{\int_0^T\sigma_s^2 ds}-\int_0^t\left(\tilde{\mu}_s+\mu_s-\frac{\tilde{\sigma}_s^2}{2}-\frac{\sigma_s^2}{2}\right)ds-\int_0^t\tilde{\sigma}_s dW_s}{\sqrt{\int_t^T\sigma_s^2 ds}}\right)\\
:= &\ \Phi\left(X_t\right):=\Phi\left(\frac{z_t}{\sqrt{n_t}}\right).
\end{aligned}
$$

We have that

$$
\begin{aligned}
da_t = &\ d\Phi\left(X_t\right)=\Phi'(X_t)dX_t+\frac{1}{2}\Phi''(X_t)d[X]_t\\
= &\ \Phi'(X_t)dX_t-\frac{1}{2}X_t\Phi'(X_t)d[X]_t\\
= &\ \Phi'(X_t)\left[dX_t-\frac{1}{2}X_t d[X]_t\right],
\end{aligned}
$$

where

$$
\begin{aligned}
dn_t = &\ -\sigma_t^2 dt,\\
dz_t = &\ -\left(\tilde{\mu}_t+\mu_t-\frac{\tilde{\sigma}_t^2}{2}-\frac{\sigma_t^2}{2}\right)dt-\tilde{\sigma}_t dW_t,\\
dX_t = &\ \frac{1}{\sqrt{n_t}}dz_t-\frac{1}{2}\frac{X_t}{n_t}dn_t\\
= &\ -\frac{1}{\sqrt{\int_t^T\sigma_s^2 ds}}\left(\tilde{\mu}_t+\mu_t-\frac{\tilde{\sigma}_t^2}{2}-\frac{\sigma_t^2}{2}\right)dt-\frac{\tilde{\sigma}_t}{\sqrt{\int_t^T\sigma_s^2 ds}}dW_t+\frac{1}{2}\frac{\sigma_t^2}{\int_t^T\sigma_s^2 ds}X_t dt,\\
d[X]_t = &\ \frac{\tilde{\sigma}_t^2}{\int_t^T\sigma_s^2 ds}dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
dX_t-\frac{1}{2}X_t d[X]_t = &\ -\frac{1}{\sqrt{\int_t^T\sigma_s^2 ds}}\left(\tilde{\mu}_t+\mu_t-\frac{\tilde{\sigma}_t^2}{2}-\frac{\sigma_t^2}{2}\right)dt-\frac{\tilde{\sigma}_t}{\sqrt{\int_t^T\sigma_s^2 ds}}dW_t\\
&-\frac{1}{2}\frac{\tilde{\sigma}_t^2-\sigma_t^2}{\int_t^T\sigma_s^2 ds}X_t dt.
\end{aligned}
$$

(c) The model of CT can be transformed into the model of CH equating the coefficients of "dt" and "dW_t"

$$-\frac{\tilde{\sigma}_t}{\sqrt{\int_t^T \sigma_s^2 ds}} = \sqrt{\frac{\beta}{T-t}}, \tag{8}$$

and

$$-\frac{1}{\sqrt{\int_t^T \sigma_s^2 ds}}\left(\tilde{\mu}_t + \mu_t - \frac{\tilde{\sigma}_t^2}{2} - \frac{\sigma_t^2}{2}\right) = h\sqrt{\frac{\beta}{T-t}}. \tag{9}$$

By setting $\tilde{\sigma}_t^2 = \sigma_t^2$ and then rearranging Equation (8) we obtain the following PDE

$$\tilde{\sigma}_t^2 \cdot \frac{T-t}{\beta} = \int_t^T \tilde{\sigma}_s^2 ds.$$

Hence for $\beta > 0$ we have $\tilde{\sigma}_t^2 = \beta(T-t)^{\beta-1} = \sigma_t^2$. Thus

$$\tilde{\sigma}_t = -\sigma_t = -\sqrt{\beta(T-t)^{\beta-1}}.$$

Applying $\tilde{\sigma}_t^2 = \sigma_t^2$ to Equation (9) and then equating (8) and (9) yields

$$\tilde{\mu}_t + \mu_t - \tilde{\sigma}_t^2 = h\tilde{\sigma}_t$$

which completes the proof. ♢

Approximation 2: A linear approximation in the model of CT leads to $\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right) = \ln\left(\frac{N-t\cdot Q_t}{(T-t)\cdot Q_t}\right)$. Now, approximating Q_t in the nominator by its expected value $\mathbb{E}\left[Q_t\right] = Q_0 e^{\mu t}$ yields

$$\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right) \approx \ln\left(\frac{N-t\cdot\mathbb{E}\left[Q_t\right]}{(T-t)\cdot Q_t}\right).$$

Theorem 3 (SDE for approximation approach 2)**.**
Using the approximation $\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right) \approx \ln\left(\frac{N-t\cdot\mathbb{E}[Q_t]}{(T-t)\cdot Q_t}\right)$, *the dynamics of the permit price in the model of CT are given by*

$$da_t = -\frac{\Phi'\left(\Phi^{-1}(a_t)\right)}{\sqrt{T-t}}\left[\left(\frac{(1+\mu t)Q_0 e^{\mu t}}{\sigma(N-t\cdot Q_0 e^{\mu t})} - \frac{1}{T-t}\right)dt + dW_t\right].$$

Proof :

Using the approximation $q_{[0,t]} = t \cdot Q_t$, we get

$$
\begin{aligned}
a_t =\ & \Phi\left(\frac{-\ln\left(\frac{N-t\cdot Q_t}{(T-t)\cdot Q_t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
=\ & \Phi\left(\frac{-\ln\left(N-t\cdot Q_t\right) + \ln(T-t) + \ln(Q_t) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
=\ & \Phi\left(\frac{-\ln\left(N-t\cdot Q_t\right) + \ln(T-t) + \ln(Q_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
=\ & \Phi\left(\frac{-\ln\left(N-t\cdot Q_t\right) + \ln(T-t) + \ln(Q_0) + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T-t}}\right) \\
=\ & \Phi\left(\frac{\Phi^{-1}(a_0)\sigma\sqrt{T} + \ln\left(\frac{N}{T}\right) - \ln\left(N-t\cdot Q_t\right) + \ln(T-t) + \sigma W_t}{\sigma\sqrt{T-t}}\right).
\end{aligned}
$$

Now, using the approximation $Q_t \approx \mathbb{E}\left[Q_t\right] = Q_0 e^{\mu t}$ and plugging in $\ln\left(N - t\cdot Q_t\right)$, yields

$$
\begin{aligned}
a_t =\ & \Phi\left(\frac{\Phi^{-1}(a_0)\sigma\sqrt{T} + \ln\left(\frac{N}{T}\right) - \ln\left(N-t\cdot Q_0 e^{\mu t}\right) + \ln(T-t) + \sigma W_t}{\sigma\sqrt{T-t}}\right) \\
:=\ & \Phi\left(X_t\right) :=\ \Phi\left(\frac{z(t)}{\sqrt{n(t)}}\right).
\end{aligned}
$$

The differential of the normalized permit price is

$$
\begin{aligned}
da_t =\ & d\Phi\left(X_t\right) = \Phi'(X_t)dX_t + \frac{1}{2}\Phi''(X_t)d[X]_t \\
=\ & \Phi'(X_t)dX_t - \frac{1}{2}X_t\Phi'(X_t)d[X]_t \\
=\ & \Phi'(X_t)\left[dX_t - \frac{1}{2}X_t d[X]_t\right],
\end{aligned}
$$

where

$$
\begin{aligned}
dn_t =\ & -\sigma^2 dt, \\
dz_t =\ & \left(\frac{(1+\mu t)Q_0 e^{\mu t}}{N - t\cdot Q_0 e^{\mu t}} - \frac{1}{T-t}\right)dt - \sigma dW_t, \\
dX_t =\ & \frac{1}{\sqrt{n_t}}dz_t - \frac{1}{2}\frac{X_t}{n_t}dn_t \\
=\ & \frac{1}{\sigma\sqrt{T-t}}\left(\frac{(1+\mu t)Q_0 e^{\mu t}}{N - t\cdot Q_0 e^{\mu t}} - \frac{1}{T-t}\right)dt - \frac{1}{\sqrt{T-t}}dW_t + \frac{1}{2}\frac{X_t}{T-t}dt, \\
d[X]_t =\ & \frac{1}{T-t}dt.
\end{aligned}
$$

Thus

$$dX_t - \frac{1}{2}X_t d[X]_t = \left(\frac{1}{\sigma\sqrt{T-t}}\frac{(1+\mu t)Q_0 e^{\mu t}}{N - t\cdot Q_0 e^{\mu t}} - \frac{1}{T-t}\right)dt - \frac{1}{\sqrt{T-t}}dW_t.$$

♢

Approximation 3: Bearing in mind that $T-t$ is an affine function and that the number of remaining permits is approximately an affine function in t, we can use the following approximation for small positive Δ

$$\frac{N - q_{[0,t+\Delta]}}{N - q_{[0,t]}}\frac{T-t}{T-(t+\Delta)} \approx 1.$$

We apply approximation 3 both to the model of CT (cf. Theorem 4) and to the model of GK model (cf. Theorem 5).

Theorem 4 (SDE for approximation approach 3)**.**
Let $\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\frac{T-t}{T-(t+\Delta)} \approx 1$ *for small positive* Δ*. Then the following difference is approximately standard normally distributed in the model of CT*

$$\frac{1}{\sqrt{\Delta}}\left(\Phi^{-1}(a_{t+\Delta})\sqrt{T-(t+\Delta)} - \Phi^{-1}(a_t)\sqrt{T-t}\right). \tag{10}$$

Proof :
By Lemma 1

$$\Phi^{-1}(a_t)\sqrt{T-t} = \frac{1}{\sigma}\cdot\left(-\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)\right).$$

Thus,

$$\begin{aligned}\Phi^{-1}(a_t)\sqrt{T-t} - \Phi^{-1}(a_{t+\Delta})\sqrt{T-(t+\Delta)}\\ &= \frac{1}{\sigma}\cdot\left(\ln\left(\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\cdot\frac{T-t}{T-(t+\Delta)}\right) - \ln\left(\frac{Q_{t+\Delta}}{Q_t}\right) + \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right)\\ &= \frac{1}{\sigma}\cdot\left(\ln\left(\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\cdot\frac{T-t}{T-(t+\Delta)}\right) - \sigma W_\Delta\right).\end{aligned}$$

Assuming $\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\cdot\frac{T-t}{T-(t+\Delta)} \approx 1$ completes the proof. ♢

Theorem 5 (Discretized SDE for the model of Grüll and Kiesel)**.**
Let $\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\frac{\frac{1}{\mu}(e^{\mu(T-t)}-1)}{\frac{1}{\mu}(e^{\mu(T-(t+\Delta))}-1)} \approx 1$ *for small positive* Δ *and let* $Z \sim N(0,1)$*. Then*

(1) The dynamics of the permit price in the model of GK are described by the following discretized SDE

$$\begin{aligned} z_t &:= \Phi^{-1}(a_{t+\Delta})\sqrt{T-(t+\Delta)} - \Phi^{-1}(a_t)\sqrt{T-t} \\ &\sim N\left(\frac{\Delta}{\sqrt{b(\mu,\sigma^2)}}\left(\mu - \frac{\sigma^2}{2} + \frac{b(\mu,\sigma^2)}{2}\right), \frac{\sigma^2\Delta}{b(\mu,\sigma^2)}\right), \end{aligned}$$

where

$$b(\mu,\sigma^2) = \frac{\mu(\mu+\sigma^2)\left(e^{2\mu+\sigma^2} - e^{\mu}\right)}{\mu e^{2\mu+\sigma^2} + \mu + \sigma^2 - (2\mu+\sigma^2)\, e^{\mu}} - 2\frac{\mu e^{\mu}}{e^{\mu}-1}.$$

(2) Let m and s^2 be the sample mean and the sample variance of the data set $\{z_t\}$. Then the parameter estimate $\hat{\sigma}^2$ is given by the solution of

$$b\left(\frac{m}{s\sqrt{\Delta}}\hat{\sigma} + \frac{1}{2}\left(1 - \frac{\Delta}{s^2}\right)\hat{\sigma}^2, \hat{\sigma}^2\right) = \frac{\Delta}{s^2}\hat{\sigma}^2, \tag{11}$$

and the estimate $\hat{\mu} := \hat{\mu}(\hat{\sigma}^2)$ is given by

$$\hat{\mu} = \frac{m}{s\sqrt{\Delta}}\hat{\sigma} + \frac{1}{2}\left(1 - \frac{\Delta}{s^2}\right)\hat{\sigma}^2. \tag{12}$$

Proof :
(a) The permit price in the model of GK is given by

$$a_t = \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) + g(T-t)}{\sqrt{h(T-t)}}\right), \tag{13}$$

where

$$g(T-t) = \ln\left(\frac{\alpha_{T-t}^2}{\sqrt{2\beta_{T-t}}}\right), \quad \text{and} \quad h(T-t) = \ln\left(\frac{2\beta_{T-t}}{\alpha_{T-t}^2}\right). \tag{14}$$

Parameters α_{T-t} and β_{T-t} are given in Equation (1) and (2), respectively. The Taylor expansion around 1 yields

$$h(\tau) = h(1) + h'(1)(\tau - 1) + \frac{1}{2}h''(\xi)(\xi - 1)^2$$

for ξ between 1 and τ. It can be shown that the error term is sufficiently small for parameter combinations (μ, σ^2) that are in scope. Furthermore it can be shown that $h(1) - h'(1) \approx 0$. Therefore, in the following we work with the approximation

$$h(T-t) \approx b(\mu,\sigma^2)(T-t),$$

where

$$b(\mu,\sigma^2) = h'(1) = \frac{\beta_1'}{\beta_1} - 2\frac{\alpha_1'}{\alpha_1}.$$

Thus

$$
\begin{aligned}
h(T-t) \approx b(\mu,\sigma^2)(T-t) &\Leftrightarrow \frac{2\beta_{T-t}}{\alpha_{T-t}^2} \approx e^{b(\mu,\sigma^2)(T-t)} \\
&\Leftrightarrow \sqrt{2\beta_{T-t}} \approx \sqrt{e^{b(\mu,\sigma^2)(T-t)}}\alpha_{T-t} \\
&\Leftrightarrow \frac{\alpha_{T-t}^2}{\sqrt{2\beta_{T-t}}} \approx \frac{\alpha_{T-t}}{\sqrt{e^{b(\mu,\sigma^2)(T-t)}}} \\
&\Leftrightarrow g(T-t) \approx \ln(\alpha_{T-t}) - \frac{1}{2}b(\mu,\sigma^2)(T-t).
\end{aligned}
$$

Inserting the approximation functions for $g(\cdot)$ and $h(\cdot)$ into Equation (13) yields

$$
\Phi^{-1}(a_t) = \frac{1}{\sqrt{b(\mu,\sigma^2)(T-t)}}\left[-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) + \ln(\alpha_{T-t}) - \frac{1}{2}b(\mu,\sigma^2)(T-t)\right],
$$

which is equivalent to

$$
\Phi^{-1}(a_t)\sqrt{T-t} = \frac{1}{\sqrt{b(\mu,\sigma^2)}}\left[-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) + \ln(\alpha_{T-t}) - \frac{1}{2}b(\mu,\sigma^2)(T-t)\right].
$$

For small positive Δ we have

$$
\begin{aligned}
&\Phi^{-1}(a_{t+\Delta})\sqrt{T-(t+\Delta)} - \Phi^{-1}(a_t)\sqrt{T-t} \\
&\quad = \frac{1}{\sqrt{b(\mu,\sigma^2)}}\left[\ln\left(\frac{N-q_{[0,t]}}{N-q_{[0,t+\Delta]}}\cdot\frac{\alpha_{T-(t+\Delta)}}{\alpha_{T-t}}\right) + \ln\left(\frac{Q_{t+\Delta}}{Q_t}\right) + \frac{\Delta}{2}b(\mu,\sigma^2)\right] \\
&\quad = \frac{1}{\sqrt{b(\mu,\sigma^2)}}\left[\ln\left(\frac{N-q_{[0,t]}}{N-q_{[0,t+\Delta]}}\cdot\frac{\alpha_{T-(t+\Delta)}}{\alpha_{T-t}}\right) + \left(\mu - \frac{\sigma^2}{2} + \frac{b(\mu,\sigma^2)}{2}\right)\Delta + \sigma W_\Delta\right].
\end{aligned}
$$

As both $N-q_{[0,t]}$ and α_{T-t} are approximately affine functions, we can use the following approximation

$$
\frac{N-q_{[0,t]}}{N-q_{[0,t+\Delta]}}\cdot\frac{\alpha_{T-(t+\Delta)}}{\alpha_{T-t}} \approx 1
$$

which completes the proof.
(b) We obtain the parameters $\hat{\mu}$ and $\hat{\sigma}^2$ by solving

$$
m = \frac{\Delta}{\sqrt{b(\hat{\mu},\hat{\sigma}^2)}}\left(\hat{\mu} - \frac{\hat{\sigma}^2}{2} + \frac{b(\hat{\mu},\hat{\sigma}^2)}{2}\right), \tag{15}
$$

$$
s^2 = \frac{\hat{\sigma}^2\Delta}{b(\mu,\hat{\sigma}^2)}. \tag{16}
$$

Solving Equation (16) for $b(\hat{\mu},\hat{\sigma}^2)$ and plugging the result into Equation (15) yields

$$\hat{\mu}(\hat{\sigma}^2) = \frac{m}{s\sqrt{\Delta}}\hat{\sigma} + \frac{1}{2}\left(1 - \frac{\Delta}{s^2}\right)\hat{\sigma}^2.$$

Inserting $\hat{\mu}(\hat{\sigma}^2)$ into Equation (16) and solving for $\hat{\sigma}^2$ completes the proof. $\diamondsuit$

Unfortunately, all the estimation methods for the models of CT and GK (cf. Theorem 3 - 5) cannot be used in practice. This can be explained as follows. All the discussed estimation methods have in common that for parameter estimation one would have to compute for $a_{t_1}, \ldots, a_{t_n}$ the values of $z_{t_i} := \Phi^{-1}(a_{t_{i+1}})\sqrt{T - t_{i+1}} - \Phi^{-1}(a_{t_i})\sqrt{T - t_i}$, calculate the empirical mean and variance of $\{z_{t_i}\}$ and then equate them to the theoretical mean and variance which is a function of the model parameters μ and σ^2. A useful estimation method should ensure that the equation can be solved for every possible combination of observed mean $m \in M \subseteq \mathbb{R}$ and variance $v \in V \subseteq \mathbb{R}^+$. In other words, the set of possible mean-variance combinations $M \times V$ should span $\mathbb{R} \times \mathbb{R}^+$. However, this is not the case as the set of possible mean-variance combinations in Theorem 3 and 4 are a line and a point, respectively. In the case of Theorem 5, it is a two-dimensional set but it does not span $\mathbb{R} \times \mathbb{R}^+$. Therefore we introduce the following reduced-form model that overcomes this difficulty.

Definition 6 (Reduced-form model of Grüll and Taschini)**.**
Assume that the permit price divided by the penalty is described by the following SDE

$$d\left(\Phi^{-1}(a_t)\sqrt{T-t}\right) = a dt + b dW_t,$$

where $a, b \in \mathbb{R}$ are the parameters of the reduced-form model under the historical measure (hereafter GT).

We employ Definition 6 for parameter estimation in the next section. For completeness, in the following Corollary we derive an SDE for the reduced-form model of GT in order to compare it to the model of CH.

Corollary 7 (SDE for reduced-form model of Grüll and Taschini)**.**
The permit price dynamics in the model of GT are given by

$$da_t = \frac{\Phi'(\Phi^{-1}(a_t))}{\sqrt{T-t}}\left[\left(a + \frac{1-b^2}{2\sqrt{T-t}}\Phi^{-1}(a_t)\right)dt + b dW_t\right].$$

Proof :

Let $X_t = \Phi^{-1}(a_t)\sqrt{T-t}$. Thus $a_t = \Phi\left(\frac{X_t}{\sqrt{T-t}}\right) := f(X_t, t)$ and

$$\begin{aligned}
f_x(x,t) &= \Phi'\left(\frac{x}{\sqrt{T-t}}\right) \cdot \frac{1}{\sqrt{T-t}}, \\
f_{xx}(x,t) &= \Phi''\left(\frac{x}{\sqrt{T-t}}\right) \cdot \frac{1}{T-t} = -\frac{x}{\sqrt{T-t}}\Phi'\left(\frac{x}{\sqrt{T-t}}\right) \cdot \frac{1}{T-t}, \\
f_t(x,t) &= \frac{1}{2}x\Phi'\left(\frac{x}{\sqrt{T-t}}\right) \cdot (T-t)^{-\frac{3}{2}}.
\end{aligned}$$

By Definition 6, we have

$$\begin{aligned}
dX_t &= adt + bdW_t, \\
d[X]_t &= b^2 dt.
\end{aligned}$$

By Ito's lemma, we obtain

$$\begin{aligned}
da_t = \;& df(X_t,t) = f_x(X_t,t)dX_t + f_t(X_t,t)dt + \frac{1}{2}f_{xx}(X_t,t)d[X]_t \\
= \;& \Phi'\left(\frac{X_t}{\sqrt{T-t}}\right) \cdot \frac{1}{\sqrt{T-t}}\left[adt + bdW_t + \frac{X_t}{2(T-t)}dt - \frac{X_t}{2(T-t)} \cdot b^2 dt\right] \\
= \;& \frac{\Phi'(\Phi^{-1}(a_t))}{\sqrt{T-t}}\left[\left(a + \frac{1-b^2}{2\sqrt{T-t}}\Phi^{-1}(a_t)\right)dt + bdW_t\right].
\end{aligned}$$

$\diamondsuit$

Remark:
The SDE for the reduced-form model of GT differs from the SDE for the model of CH by the additional term $\frac{1-b^2}{2\sqrt{T-t}}\Phi^{-1}(a_t)dt$.

4 Empirical analysis

In 2005 European policy makers launched the EU ETS, the world's largest emission trading system which covers approximately 50% of the CO_2 emissions in the European Union. The EU ETS consists of three different phases. Phase I lasted until the end of 2007. Phase II started in 2008 and ends in 2012. A third phase will start in 2013. Due to bankability restrictions between phase I and II, it is necessary to treat the price series of each phase separately - see Alberola and Chavallier [1]. As the futures market is more liquid than the spot market, in what follows we perform our model calibration analysis with price series of futures contracts maturing in December 2007 and December 2012, respectively. In the first phase the price of emission permits is characterized by a very high volatility level. The significant market correction between the end of April and the beginning of May 2006 (see Figure 1) occurred when emission data for the year 2005 became public showing that there was an overall overestimation of offending emissions. A long-lasting futures December 2007 price decrease, characterized by a smaller volatility, started in August 2006. Such a price behaviour is typical for permit prices at the end of a compliance period. This has to do with the fact that at compliance time the permit price can only take the values zero (overallocation) or the penalty fee (permit shortage). As the reduced-form models also have this property one should expect that they excel in capturing the observed price dynamics at the end of a compliance period. In order to test this hypothesis we split up the futures December 2007 price series into two parts. We take the period of the crash as a cutting point. Prices observed during the crash (i.e. 15 trading days) are not included into our analysis. Another effect that can be observed at the end of the comliance period is that from May, 10th 2007 transaction volumes are very low and the permit price hovers below 0.30 € remaining at the same price level for several consecutive days. We consider this special effect by performing our analysis both on the full post-crash price series and on the series that is truncated on May, 10th 2007. Finally, for phase II we consider futures contracts with maturity December 2012 from January, 2nd 2008 until August, 31th 2009. The futures permit price in this period exhibits a lower volatility level hinting at a relatively more mature market. As observable in Figure 1, futures December 2012 prices range from 10 € to 35 €, peaking on July, 1st 2008 at 34.38 €. So, in summing, we analyze the following four data series:

(1) pre-crash phase I (22 April 2005 - 24 April 2006)
(2) post-crash phase I (15 May 2006 - 17 December 2007)
(3) truncated post-crash phase I (15 May 2006 - 10 May 2007)
(4) phase II (2 January 2008 - 31 August 2009)

Besides comparing performances of the reduced-form models of CH and GT, we calibrate other continuous-time stochastic processes and undertake an extensive model comparison. In particular, we restrict ourselves to widely known stochastic processes, such as geometric Brownian motion (GBM) and normal Inverse Gaussian (NIG). The last is an extensively used and more complex process that overcomes some of the drawbacks of the GBM. For instance, it captures the presence of fat tails.

Because residuals of the reduced-form models and of the GBM are normally distributed, whereas residuals of the NIG process are not normally distributed, we consider two different type of analysis. We first run normality tests to all models with normally distributed residuals providing an investigation of the goodness-of-fit of reduced-form models and the GBM (cf. Table 2-5). Second, we assess in-sample performances of NIG, GBM, the model of CH and the model of GT by comparing Q-Q-plots (cf. Figure 2-5) and computing the

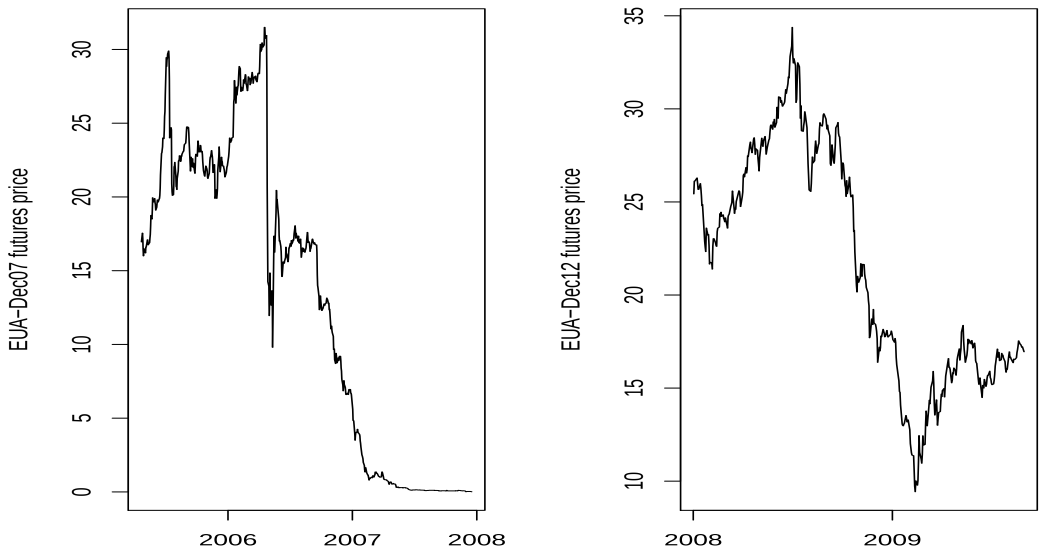

Fig. 1. Left: EUA-Dec07 futures price (22 April 2005 - 17 December 2007), right: EUA-Dec12 futures price (2 January 2008 - 31 August 2009)

Kolmogorov-Smirnov-distance (cf. Table 1).
As expected, our empirical analysis shows that reduced-form models exhibit their strength at the end of a compliance period. Taking the full post-crash price series into account the reduced-form models outperform both GBM and NIG (cf. Figure 3 and Table 1 and 3). However, the Q-Q-plots in Figure 4 reveal that even reduced-form models cannot completely capture the price dynamics in this particular period. Excluding the special effect of very high volatility due to prices very close to zero and low trading volume at the very end of the first compliance period (after May, 10th 2007) we get a slightly different picture. Reduced-form models still outperform GBM but perform worse than the more complex process NIG (cf. Figure 4 and Table 1 and 4). At the beginning of a compliance period the price dynamics are by far captured better by NIG than the tailor-made reduced-form models. Compared to GBM, the reduced-form models perform slightly worse at the beginning of the first phase (cf. Table 2) and similarly at the beginning of the second phase (cf. Table 5). Finally, the two competing reduced-form models of GT and CH have a similar performance whereby the model of GT slightly outperforms the model of CH at the very end of the first compliance period (cf. Table 2-5). Summarizing, reduced-form models perform relatively well at the end of a compliance period compared to standard stochastic processes. However, they are clearly outperformed by complex standard stochastic processes, especially, at the beginning of the two compliance periods.

	NIG	GBM	Carmona & Hinz	Grüll & Taschini
Phase 1 - Pre-Crash Period				
KS-Distance	**0.0321**	0.0928	0.1207	0.1179
Phase 1 - Post-Crash Period				
KS-Distance	0.1716	0.2188	0.1645	**0.1037**
Phase 1 - Post-Crash Period (truncated)				
KS-Distance	**0.0683**	0.144	0.0951	0.0994
Phase 2				
KS-Distance	**0.0257**	0.0757	0.0816	0.0785

Table 1
Comparison of goodness-of-fit.

5 Conclusions

We derive three estimation methods for the equilibrium models proposed by Chesney and Taschini [7] (CT) and Grüll and Kiesel [11] (GK) for modeling the price of emission permits. The resulting estimation methods for the models of CT and GK cannot be used in practice. This has to do with the fact that the obtained SDEs do not possess sufficient free parameters for model-calibration and, therefore, are not flexible enough to capture the historical permit price evolution. We propose a new reduced-form model (hereafter denoted by GT) based on the full equilibrium models of CT and GK. Furthermore, we show how the model of CT with time-dependent emission rate can be transformed into the reduced-form model proposed by Carmona and Hinz [5] (CH).

Using futures prices in the EU ETS with maturity December 2007 and December 2012, for the first time in the literature we calibrate reduced-form models and assess the in-sample performances of the models of CH and GT. With the aim of providing a comprehensive comparison among potentially competing models, we also calibrate and compare two quite popular continuous-time stochastic processes (GBM and NIG). In a perfect competitive equilibrium framework with no-banking options, futures permit prices are characterized by the fact that they tend to either zero or the penalty fee at the end of a compliance period. As reduced-form models capture this characteristic, we split up the permit price series in order to analyze the performance both at the beginning and at the end of a compliance period. In the current price-evolution, we observe that reduced-form models perform relatively well at the end of a compliance period compared to standard stochastic processes. However, they are clearly outperformed by complex standard stochastic processes such as NIG, especially, at the beginning of the two compliance periods. GBM and reduced-form models perform similarly at the beginning of a compliance period. However, reduced-form models describe the price dynamics at the end of the first compliance period much better than GBM. Finally, the two competing reduced-form models of GT and CH have a similar performance whereby the model of GT slightly outperforms the model of CH at the very end of the first compliance period.

The evaluation of the price of emission permits in the coming years will show whether, in a more mature permit market, complex standard stochastic processes such as NIG still outperform reduced-form models that take into account peculiar characteristics of permit markets.

6 Appendix

The residuals of GBM and the reduce-form models of CH and GT are all standard normally distributed. Therefore we can apply normality tests to the log-returns in the case of GBM, to the data transformed according to the discretized version of Equation (7) in the case of the model of CH and to the data transformed according to the discretized version of Definiton 6 in the case of the model of GT. We omit the usual footnotes concerning the significance of the normality tests as the null hypothesis that the data is normally distributed is rejected throughout at the 5% significance level. The tables show the test statistics of the performed normality tests. The most favourable test statistic for normality (i.e. the lowest) is marked bold in each row.

Normality test	GBM	Carmona & Hinz	Grüll & Taschini
Kolmogorov-Smirnov	**0.0928**	0.1207	0.1179
Anderson-Darling	**5.2260**	7.5697	7.1298
Pearson	**39.594**	67.106	67.255
Jarque-Bera	**1734.8**	3458.7	2792.4
Cramer-von-Mises	**0.8326**	1.2122	1.1363

Table 2
Comparison of goodness-of-fit (Pre-Crash).

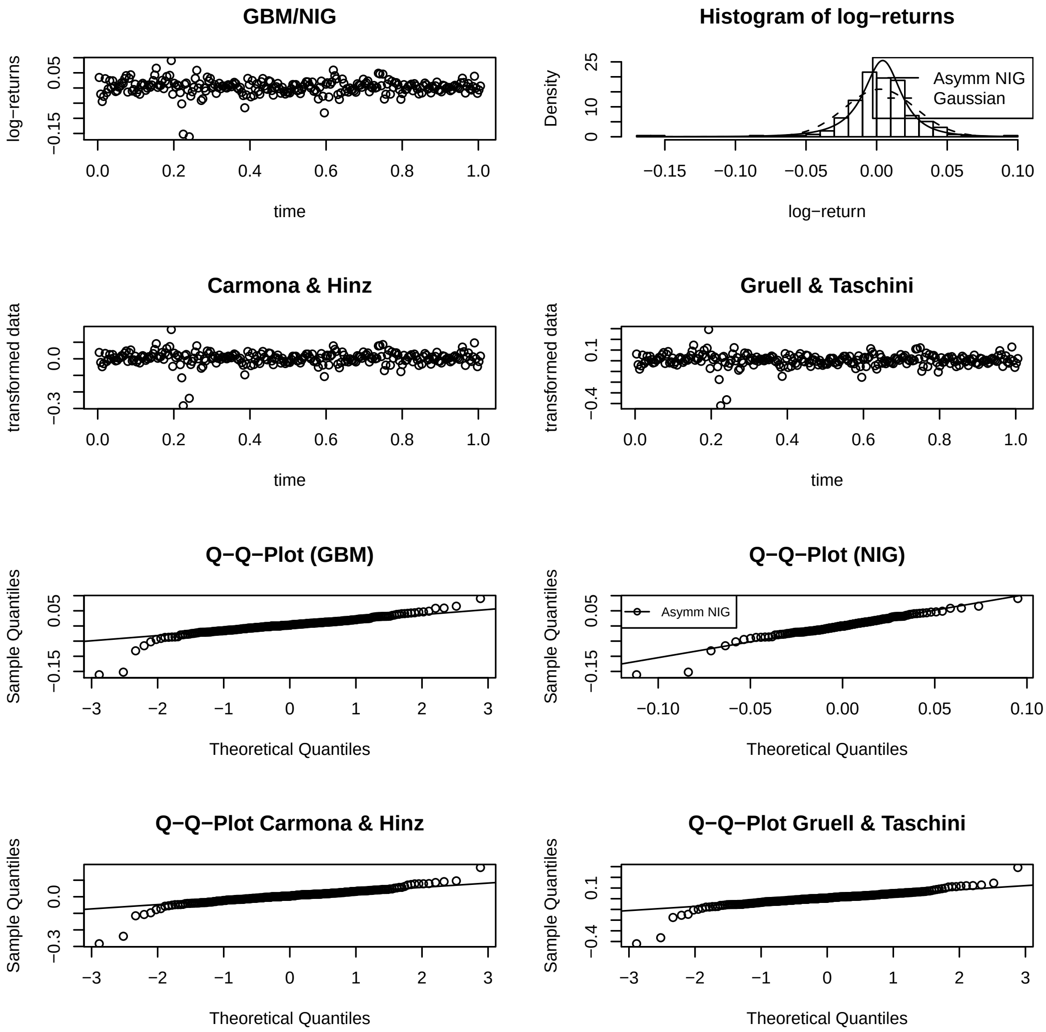

Fig. 2. Log-returns, transformed data and Q-Q-plots of different models for pre-crash-period

Normality test	GBM	Carmona & Hinz	Grüll & Taschini
Kolmogorov-Smirnov	0.2188	0.1645	**0.1037**
Anderson-Darling	∞	13.213	**9.800**
Pearson	1048.5	689.3	**136.15**
Jarque-Bera	50059	406	**233**
Cramer-von-Mises	8.6221	2.6040	**1.7628**

Table 3
Comparison of goodness-of-fit (Post-Crash).

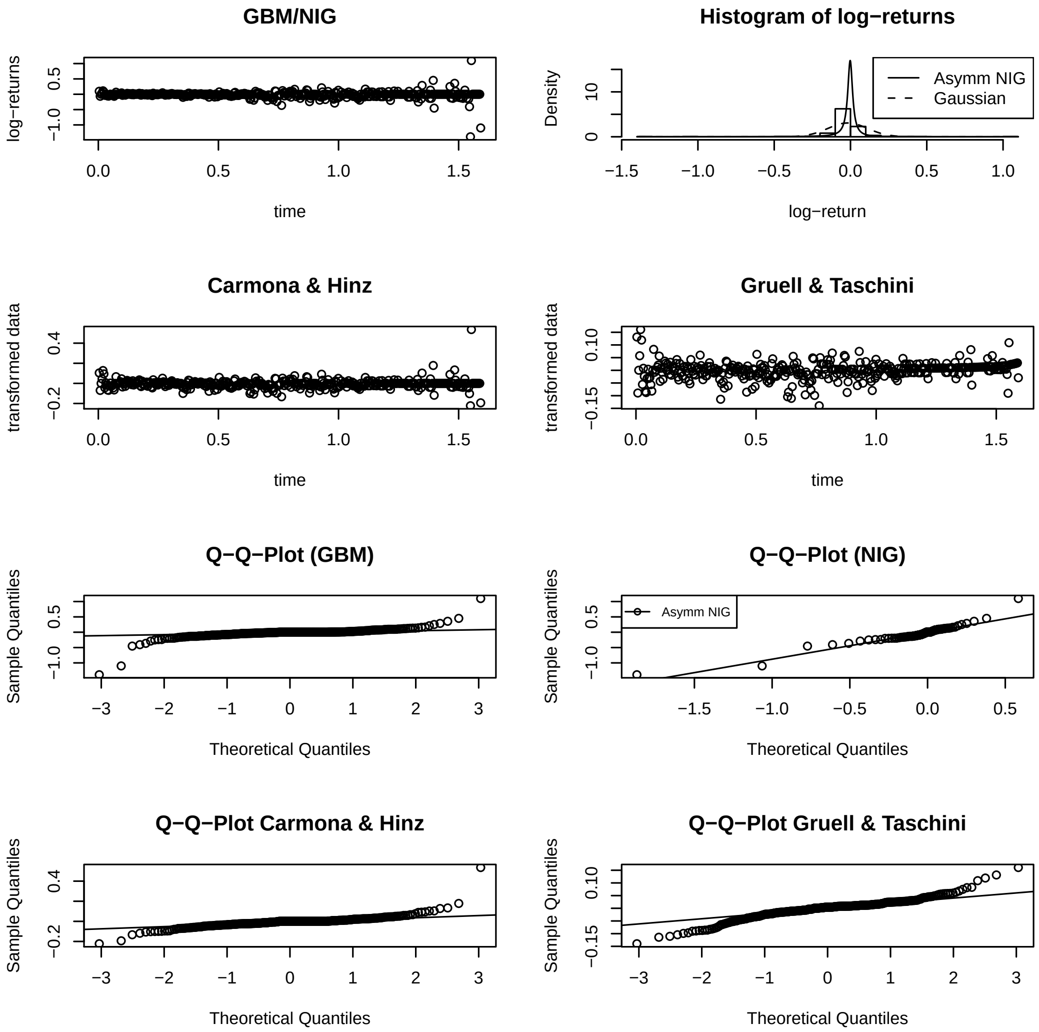

Fig. 3. Log-returns, transformed data and Q-Q-plots of different models for post-crash-period

Normality test	GBM	Carmona & Hinz	Grüll & Taschini
Kolmogorov-Smirnov	0.1440	**0.0951**	0.0994
Anderson-Darling	10.581	**3.887**	4.277
Pearson	171.93	113.28	**58.53**
Jarque-Bera	387.94	82.39	**78.14**
Cramer-von-Mises	2.0310	**0.7166**	0.7889

Table 4
Comparison of goodness-of-fit (Post-Crash truncated).

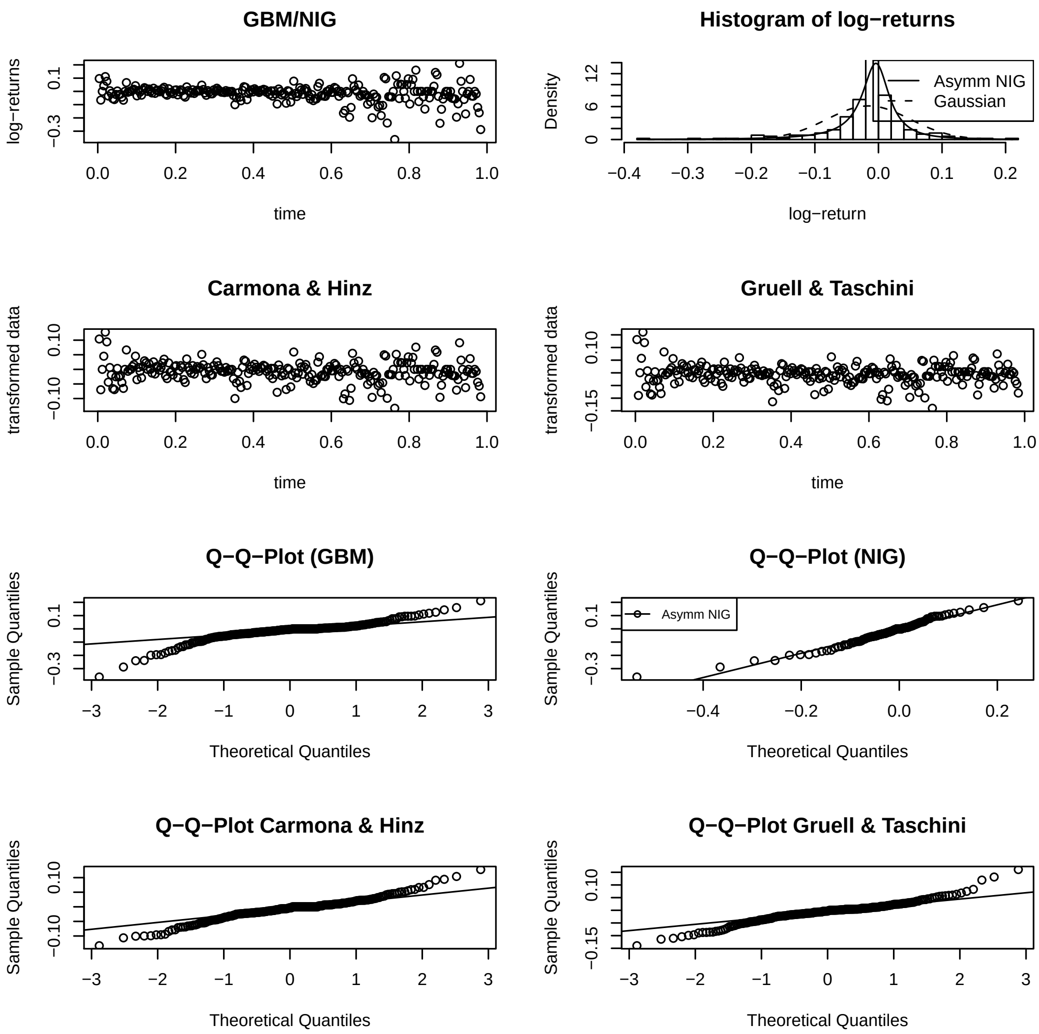

Fig. 4. Log-returns, transformed data and Q-Q-plots of different models for post-crash-period (truncated)

Normality test	GBM	Carmona & Hinz	Grüll & Taschini
Kolmogorov-Smirnov	**0.0757**	0.0816	0.0785
Anderson-Darling	3.2396	3.3747	**3.0556**
Pearson	46.741	44.896	**43.377**
Jarque-Bera	**72.644**	212.838	140.951
Cramer-von-Mises	0.5395	0.5381	**0.4884**

Table 5
Comparison of goodness-of-fit (Second Phase).

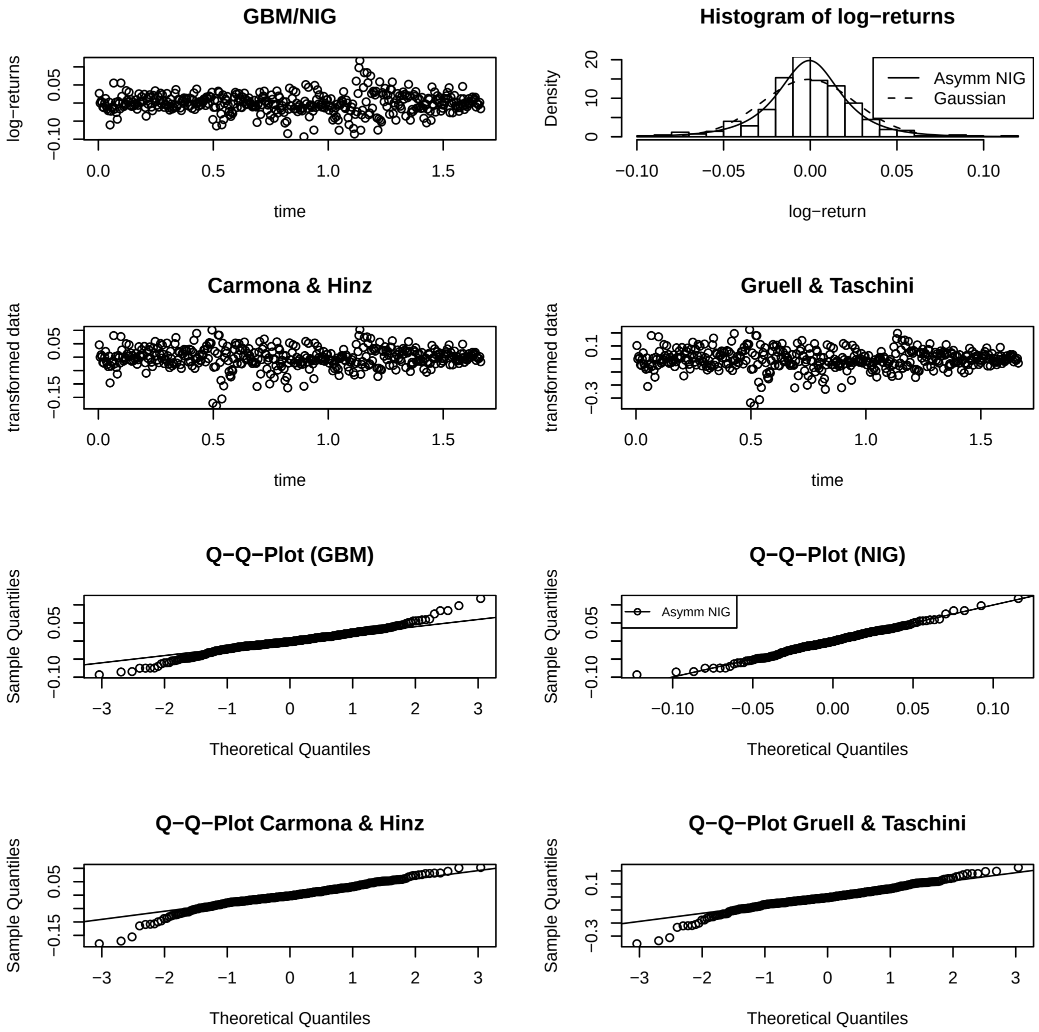

Fig. 5. Log-returns, transformed data and Q-Q-plots of different models for second phase

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